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Instability of unsteady flows or configurations. Part 2. Convective instability

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The formation of convective cells in a fluid between two horizontal rigid boundaries with time-periodic temperature distribution is studied by the use of the Floquet theory. Numerical results for the critical Rayleigh number are given for a Prandtl number of 0·73 (air) and for various values of the frequency and magnitude of the primary temperature oscillation. Some numerical results for a Prandtl number of 7·0 (water) are also given. The most striking feature of the results is that the disturbances (or convection cells) oscillate either synchronously or with half frequency.

1. Introduction

In numerous stability problems in mechanics, oscillation of the basic state has been found sometimes to have a stabilizing and sometimes a destabilizing effect, the latter often being associated with the reduction of the problem to Mathieu's equation. Some pertinent references in the field of fluid mechanics are Benjamin & Ursell (1954), Rosenblat (1968), Yih (1968), and the experimental work of Donnelly (1964).

The present work concerns Bénard convection with a time-dependent basic state. Three recent papers on that subject are those by Gresho & Sani (1970), who studied the effect of time-variable gravity on thermal convection, and Venezian (1969), on convection in an unsteady temperature field when the amplitude of the unsteady part is assumed small, and that of Rosenblat & Herbert (1970) on the same problem at low modulation frequency, but with the modulation amplitude not small. In this paper we study the marginal stability of a fluid layer with a symmetric temperature gradient which contains an unsteady part. Neither the amplitude nor the frequency of this unsteady part is assumed small. Solutions are obtained by a method of expansion in orthogonal functions akin to that of Chandrasekhar (1954). A similar investigation has been very recently reported by Rosenblat & Tanaka (1971); they use a slightly different basic temperature field, and Galerkin's method. The most striking feature of our results is that the disturbances, at neutral stability at least (and probably at instability), are either synchronous with the primary temperature field or have half its frequency. A similar but less extensive result was obtained by Gresho & Sani.

2. Primary temperature distribution

Consider a layer of fluid between two fixed plates, one at $x_3 = +\frac{1}{2}d$ and the other at $x_3 = -\frac{1}{2}d$, x_1, x_2 and x_3 being Cartesian co-ordinates with x_3 measured in the direction of the vertical from the plane mid-way between the plates. The temperature of the upper plate is kept at $T_1 + T_2 \cos \omega_* t$, and that at the lower plate is maintained at $T_0 - T_2 \cos \omega_* t$, t being the time and ω_* being equal to 2π times the frequency of the periodic temperature fluctuation at the lower plate.

Define the following dimensionless variables:

$$\left.\begin{aligned} \tau = t\kappa/d^2, \quad (x,y,z) = (1/d)(x_1, x_2, x_3), \\ \omega = \omega_* d^2/\kappa, \quad \theta = (\overline{T} - T_1)/(T_0 - T_1), \end{aligned}\right\} \tag{1}$$

where κ is the thermal diffusivity. Then the dimensionless equation governing the distribution of the primary (or mean) temperature T is

$$\partial\theta/\partial\tau = \partial^2\theta/\partial z^2 \tag{2}$$

and the boundary conditions are

$$\theta = 1 - b\cos\omega\tau \quad \text{at} \quad z = -\tfrac{1}{2}, \tag{3}$$

$$\theta = b\cos\omega\tau \quad \text{at} \quad z = \tfrac{1}{2}, \tag{4}$$

with

$$b = T_2/(T_0 - T_1). \tag{5}$$

The solution of (2) with the boundary conditions (3) and (4) is

$$\theta = -\tfrac{1}{2} - z + bF(z,\tau), \tag{6}$$

where

$$F(z,\tau) = (B\cos\omega\tau - C\sin\omega\tau)\sinh\beta z\cos\beta z - (C\cos\omega\tau + B\sin\omega\tau)\cosh\beta z\sin\beta z, \tag{7}$$

with

$$B = -\sinh\beta'\cos\beta'/(\sinh^2\beta' + \sin^2\beta'), \quad C = -B\tan\beta'\coth\beta', \tag{8}$$

and

$$\beta = (\tfrac{1}{2}\omega)^{\frac{1}{2}}, \quad \beta' = \tfrac{1}{2}\beta. \tag{9}$$

If ρ_0 is the density at temperature T_0 and the prevailing pressure, the density ρ at any temperature not too different from T_0 is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{10}$$

where α is the thermal expansion coefficient of the fluid, which is assumed incompressible. The Boussinesq approximation will be made.

3. Differential system governing stability

We shall now disturb the fluid and see whether the disturbance will grow. The temperature disturbance will be denoted by T' and the velocity by (u_1, u_2, u_3). Then (see, for instance, Pellew & Southwell 1940), with the substitutions

$$u_3 = (\kappa/d) f(x,y)\, w(z,\tau), \tag{11}$$

$$T' = (T_0 - T_1) f(x,y)\, \theta(z,\tau), \tag{12}$$

where

$$f_{xx} + f_{yy} + a^2 f = 0, \tag{13}$$

a being the wavenumber of the Bénard cells, the linearized Boussinesq equations reduce to

$$\left[\frac{1}{\sigma}\frac{\partial}{\partial\tau}-(D^2-a^2)\right](D^2-a^2)\,w = -Ra^2\theta, \tag{14}$$

$$\left[\frac{\partial}{\partial\tau}-(D^2-a^2)\right]\theta = -[-1+bF'(z-\tau)]\,w, \tag{15}$$

in which $R = g\alpha(T_0-T_1)\,d^3/\kappa\nu$ = Rayleigh number, $\sigma = \nu/\kappa$ = Prandtl number, $D = \partial/\partial z$ and $F'(z,\tau) = DF(z,\tau)$. The boundary conditions are

$$w = 0 = Dw \quad \text{at} \quad z = \pm\tfrac{1}{2}, \tag{16}$$

$$\theta = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{17}$$

4. Method of approach

We note, first of all, that $F(z,\tau)$ is an odd function of z and hence $F'(z,\tau)$ is an even function of z. Inspection of (14)–(17) reveals that the eigenfunctions can be divided into two categories: those which are even functions of z and those which are odd functions of z. All previous investigations of convection cells have shown that disturbances corresponding to even eigenfunctions and hence having an *odd* number of cells in the z direction are more unstable than those corresponding to odd functions. Of these the single-celled disturbance is the most unstable. We therefore investigate only the stability of disturbances with even eigenfunctions.

Keeping in mind the boundary conditions on θ and w, we can expand θ in a series in $\cos(2n+1)\pi z$ and w in a series in $\phi_n(z)$, which is defined by the system

$$\left.\begin{aligned}(D^2-a^2)^2\phi_n &= \cos(2n+1)\pi z,\\ \phi_n(-\tfrac{1}{2}) = 0 = D\phi_n(-\tfrac{1}{2}),&\quad \phi_n(\tfrac{1}{2}) = 0 = D\phi_n(\tfrac{1}{2}).\end{aligned}\right\} \tag{18}$$

The solution of (18) is

$$\phi_n = P_n\cosh az + Q_n z\sinh az + c_n^2\cos[(2n+1)\pi z], \tag{19}$$

where

$$\left.\begin{aligned}P_n &= -(-1)^n(2n+1)\pi c_n^2\sinh(\tfrac{1}{2}a)/(a+\sinh a),\\ Q_n &= (-1)^n 2(2n+1)\pi c_n^2\cosh(\tfrac{1}{2}a)/(a+\sinh a),\\ c_n &= 1/[(2n+1)^2\pi^2+a^2].\end{aligned}\right\} \tag{20}$$

We now expand θ and w as follows:

$$\theta = \sum_{n=0}^{\infty} B_n(\tau)\cos[(2n+1)\pi z], \tag{21}$$

$$w = \sum_{n=0}^{\infty} A_n(\tau)\,\phi_n(z). \tag{22}$$

Substituting (21) and (22) into (14) and (15), multiplying the resulting equations by $\cos[(2n+1)\pi z]$ and integrating between $z = \pm\frac{1}{2}$ gives

$$\frac{2}{\sigma}\sum_{n=0}^{\infty}\zeta_{mn}A_n'(\tau)-A_m(\tau) = -Ra^2B_m(\tau) \quad (m = 0,1,2,\ldots), \tag{23}$$

$$B_m'(\tau)+\frac{1}{c_m}B_m(\tau) = -2\sum_{n=0}^{\infty}(c_m\zeta_{mn}+b\lambda_{mn}\cos\omega\tau+b\xi_{mn}\sin\omega\tau)\,A_n(\tau) \quad (m = 0,1,2,\ldots), \tag{24}$$

where

$$\zeta_{mn} = (-1)^{m+n}\, 8a(2n+1)(2m+1)\,\pi^2 c_n^2 c_m \cosh^2(\tfrac{1}{2}a)/(a+\sinh a) - \tfrac{1}{2}c_n\,\delta_{mn}, \quad (25)$$

δ_{mn} being the Kronecker delta, and λ_{mn} and ξ_{mn} are defined by

$$\lambda_{mn}\cos\omega\tau + \xi_{mn}\sin\omega\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} F'(z,\tau)\cos[(2m+1)\pi z]\,\phi_n(z)\,dz. \quad (26)$$

When expanded, (26) gives

$$\lambda_{mn} = \beta\int_{-\frac{1}{2}}^{\frac{1}{2}} [(B-C)\cosh\beta z\cos\beta z - (B+C)\sinh\beta z\sin\beta z] \times \cos[(2m+1)\pi z]\,\phi_n(z)\,dz, \quad (26a)$$

$$\xi_{mn} = -\beta\int_{-\frac{1}{2}}^{\frac{1}{2}} [(B+C)\cosh\beta z\cos\beta z + (B-C)\sinh\beta z\sin\beta z] \times \cos[(2m+1)\pi z]\,\phi_n(z)\,dz. \quad (26b)$$

We shall now study truncated versions of the infinite set of equations (23) and (24) determining $A_n(\tau)$ and $B_n(\tau)$. If A_n and B_n remain small for all times the fluid is stable; if they grow with time in the long run (that is, cycle after cycle of period $2\pi/\omega_*$), the fluid is unstable. We shall attempt only to determine the Rayleigh number R for neutral stability.

5. Analysis

Since the coefficients in (23) and (24) are either constant or periodic functions of τ with (dimensionless) period $\tau_0 = 2\pi/\omega$, the Floquet theory applies. (See Ince 1944, pp. 381–382; Coddington & Levinson 1955, pp. 78–81.) The outstanding result of the Floquet theory is that the solution must have the form $e^{\mu_1\tau}P(\tau)$, where $P(\tau)$ is either a periodic function of τ with period τ_0 or a sum of terms each of which is the product of a polynomial in τ (in particular a constant) and such a periodic function. In all cases the vanishing of the real part μ_{1r} of μ_1 evidently marks the stability boundary. Our aim is to determine the conditions under which $\mu_{1r} = 0$.

If μ_{1i}, the imaginary part of μ_1, is zero, then the disturbance is synchronous with the unsteady part of the mean temperature field. If $\mu_{1i}\tau_0$ is equal to π or $-\pi$, then the disturbance has frequency half that of the unsteady mean temperature field. Numerical calculation will show that the most unstable disturbance either is synchronous or has half the frequency of the mean temperature. Values of $\mu_{1i}\tau_0$ other than zero and $\pm\pi$ were searched for, though not exhaustively, but none were found.

We now consider the $2M$ equations obtained by putting $m = 0, 1, \ldots, M-1$ in (23) and (24). For convenience of exposition, let us denote $2M$ by N. In principle the N equations with N unknowns can be combined into one differential equation of order N, with N independent solutions, which we shall denote by $G_n(\tau)$ $(n = 1, 2, \ldots, N)$. Let us give $G_n(\tau)$ the property

$$G_n^{(k-1)}(0) = \delta_{nk}, \quad (27)$$

where

$$G_n^{(k)}(\tau) = d^k G_n(\tau)/d\tau^k. \quad (28)$$

Then, since the coefficients of the Nth-order differential equation have period τ_0,

$$G_n(\tau+\tau_0) = \sum_{m=1}^{N} a_{nm} G_m(\tau), \tag{29}$$

and we have

$$a_{nm} = G_n^{(m-1)}(\tau_0). \tag{30}$$

If we seek a solution $U(\tau)$ with the property

$$U(\tau+\tau_0) = sU(\tau), \tag{31}$$

we can write

$$U(\tau) = \sum_{m=1}^{N} b_m G_m(\tau). \tag{32}$$

Substituting (29) and (32) into (31), we have the secular equation

$$\det\{G_n^{(k-1)}(\tau_0) - s\delta_{nk}\} = 0, \tag{33}$$

which determines s. It is evident that

$$s = e^{\mu_1 \tau_0}. \tag{34}$$

We thus have a means of determining μ_1.

For the first approximation we retain only the first term in expansions (21) and (22); equations (23) and (24) then contain only A_0 and B_0. By eliminating B_0 between them, we obtain

$$A_0''(\tau) - \left(\frac{\sigma}{2\zeta_{00}} - \frac{1}{c_0}\right) A_0'(\tau) - \frac{\sigma}{\zeta_{00}}\left[\frac{1}{2c_0} + Ra^2(c_0\zeta_{00} + b\lambda_{00}\cos\omega\tau + b\xi_{00}\sin\omega\tau)\right] A_0(\tau) = 0. \tag{35}$$

For the second approximation we have four simultaneous first-order equations or two second-order simultaneous equations, which we shall not present, because they are rather lengthy.

If desired, higher order approximations can be carried out in much the same way. Numerical computations have been carried out for a Prandtl number† $\sigma = 0{\cdot}73$ (air) but to the second approximation only. We are also in possession of asymptotic solutions of (35) and of the two second-order equations in the second approximations for large values (>100) of the Prandtl number σ. However, for these high values the mean temperature varies a good deal with the co-ordinate z and the validity of a mere second approximation becomes questionable. For this reason we do not present these solutions.

6. The case $T_1 = T_0$

Before we present the results of numerical calculation we wish to discuss the case $T_1 = T_0$, which has also been investigated. The primary temperature distribution is still governed by (2), except that T_2 is now used as the temperature scale instead of $T_0 - T_1$, so that

$$\theta = (T - T_0)/T_2. \tag{36}$$

With the boundary temperatures maintained as before (see the beginning of § 2),

$$\theta = \pm\cos\omega\tau \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{37}$$

† We also have some results for $\sigma = 7{\cdot}0$ (water). See table 2.

The solution of (2) with (37) is

$$\theta = F(z,\tau), \tag{38}$$

with $F(z,\tau)$ given by (7).

The differential equations governing stability are still (14) and (15), except that the right-hand side of (15) is to be replaced by $-F'(z,\tau)\,w$, and the Rayleigh number R is now defined by

$$R = g\alpha T_2 d^3/\kappa\nu. \tag{39}$$

We again use the expansions (21) and (22), and once more obtain (23). Equation (24) is now replaced by

$$B'_m(\tau)+\frac{1}{c_m}B_m(\tau) = -2\sum_{n=0}^{\infty}(\lambda_{mn}\cos\omega\tau+\xi_{mn}\sin\omega\tau)\,A_n(\tau) \quad (m = 0,1,2,\ldots), \tag{40}$$

where ζ_{mn}, c_m, λ_{mn} and ξ_{mn} are again as given in §4.

The equations for the various stages of approximation are very similar to those for the case $T_0 \neq T_1$, and the rest of the analysis is identical to that given in §5 for the case $T_0 \neq T_1$.

7. Discussion of results

The numerical computation was done by the Runge–Kutta method using 'double precision', with a UNIVAC 1108 computer. The step size h chosen was small enough for μ_1 in (34) to be accurate to the third significant digit at least.

Since for the case $T_0 = T_1$ there is one less parameter to consider, i.e. the parameter b does not appear, it is possible to obtain a relationship between the critical Rayleigh number R_c (note that R is based on T_2, not $2T_2$) and the corresponding critical wavenumber a_c in terms of functions of the dimensionless frequency ω of the oscillating primary temperature field. The critical Rayleigh number is the minimum value of R as a function of the wavenumber a for a fixed value of ω. Figure 1 shows R_c and a_c as functions of ω. In this figure the symbol S signifies 'synchronous' and the symbol H 'half-frequency'. On and above an S curve the disturbances are synchronous, and on and above an H curve the disturbances have the frequency $\frac{1}{2}\omega$. We have found that each of the cusps in the R_c–ω curve is really the intersection of an H curve with an S curve, both of which can be continued beyond the intersection. Thus in the area above an H curve there are also synchronous disturbances, but disturbances with half-frequency can be expected to be more unstable. Similarly, above an S curve there are disturbances with half-frequency, but synchronous disturbances are more unstable. The critical wavenumber appears to be discontinuous from the a_c–ω curve only because we do not continue the S curves and the H curves beyond their intersections. The a_c–ω curves are also composed of S curves and H curves.

Below $\omega = 6$ the arcs corresponding to the S curves and H curves, as indicated by the computed points, seem to have rather low amplitudes. That is to say, they do not sag much below the curve passing through their mean positions. For this reason we use a broken line to indicate the mean position of computed points only. The exact form of the R_c–ω curve must be very near this broken line.

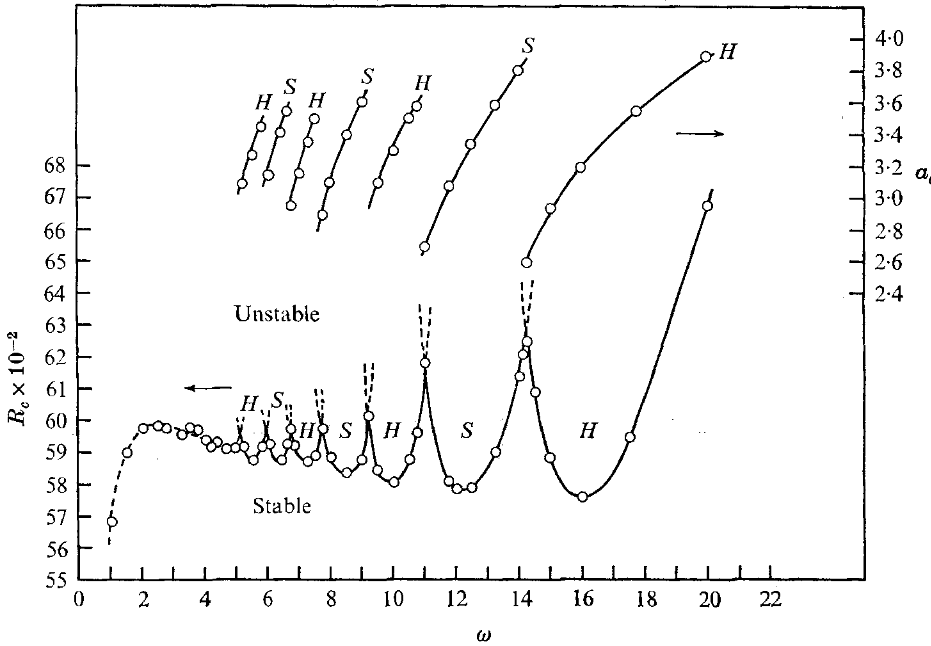

FIGURE 1. Variation of R_c and a_c with ω for $\sigma = 0{\cdot}73$ (air) and $T_0 = T_1$. S signifies 'synchronous' and H signifies 'half-frequency', for the curve below.

We note that the existence of synchronous and half-frequency disturbances was already indicated in figure 4 of the paper of Gresho & Sani (1970), in which, incidentally, the vertical co-ordinate should be $R_c \times 10^{-3}$ instead of $R_c \times 10^{-5}$ (which is merely a misprint).

At high values of ω, the primary temperature field has at any instant many 'waves' in the z direction, and the validity of an approximation taking into account only A_0, A_1, B_0 and B_1 becomes questionable. However, up to $\omega = 20$ (see figure 1) examination of the expression for $F(z, \tau)$ shows that the primary temperature has about *one* wave in the direction of z, since

$$\beta/\pi = (\omega/2\pi^2)^{\frac{1}{2}} = 1 \quad \text{for} \quad \omega = 20.$$

Thus $\phi_1(z)$ and $\phi_2(z)$ are quite sufficient to give a good approximation to the eigenfunction.

We now turn to the case $T_0 \neq T_1$. Figure 2 shows the variation of the critical Rayleigh number R_c with b for $\sigma = 0{\cdot}73$ (air) and $\omega = 5$. We use a dashed line to show the results of the first approximation and a solid line to show the results of the second approximation. We have marked with S (synchronous) and H (half-frequency) the first two arcs only, to avoid confusion, since the positions of the S curves and H curves shift from the first to the second approximation. For each approximation the S curves alternate with the H curves.

We note that in the mean the results of the first and second approximations do not differ very much, although the shift of the cusps is very evident. It is interesting that, as b increases from zero, at first (for small b) the unsteady part

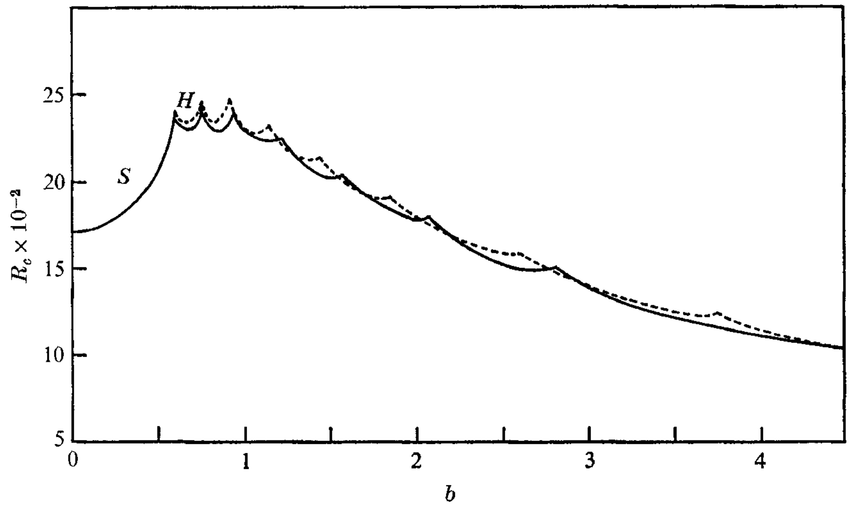

FIGURE 2. Variation of R_c with b at $\sigma = 0{\cdot}73$ and $\omega = 5$. – – –, first approximation; ——, second approximation. S signifies 'synchronous' and H signifies 'half-frequency', for the curve below. S and H alternate for the curves to the right.

of the primary temperature field is stabilizing. This agrees largely with the findings of Venezian (1969), who found a (very weak) destabilizing effect of the unsteady part of the primary temperature only in one special case of σ, and treated both boundaries as 'free'. It also agrees with Donnelly's (1964) findings for the related problem of Taylor vortices that oscillation of one cylinder can only stabilize the Couette flow. Beyond $b = 1$, however, the effect of this part is generally destabilizing.

Figure 3 is similar in every respect to figure 2, with the only difference that the R is for neutral stability at $a = 3{\cdot}117$ instead of $a = a_c$. Hence the R is a little more than the R_c in figure 2. The fact that they do not differ by very much shows that a_c is never very different from 3.

For both figures 2 and 3, the value of R_c or R at $b = 0$ is 1715·08 for the first approximation and 1707·94 for the second, both of which correspond to $a = 3{\cdot}117$. These figures agree, as they should, with Chandrasekhar's (1961) values. With this in mind it is interesting to mention that Gresho & Sani (1970) used Galerkin's method and obtained (for their problem and for $\omega = 0$, which corresponds to the Bénard problem and to $b = 0$ in our study) for R_c the value 1825 at $a = 3{\cdot}117$ with one trial function, and the value 1710·1 at $a = 3{\cdot}117$ with five trial functions. The 'exact' value is 1707·8.

In view of the fact that the R_c–b or R–b curves in figures 2 and 3 are smooth for $b < 0{\cdot}6$, we have computed the R_c values for various values of $b \leqslant 4{\cdot}5$, for values of ω other than 5, and for $\sigma = 0{\cdot}73$ (air). The results are given in table 1. For $\sigma = 7$ (water) table 2 gives R_c and a_c for $\omega = 10$ and various values of b. Both tables show that in the range of b indicated the unstable part of the primary temperature field is stabilizing.

It remains to compare our results with those of Rosenblat & Tanaka (1971). If one compares their figures 3 and 4 with our figures 2 and 3, it appears that

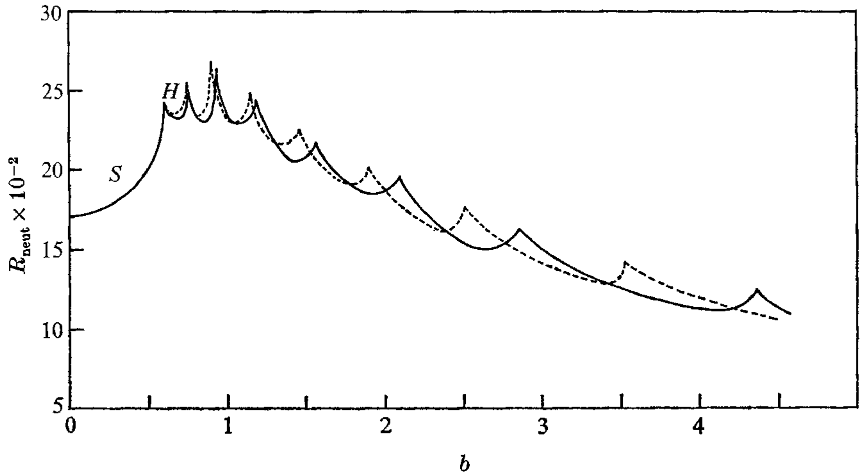

FIGURE 3. Variation of R at neutral stability with b at $\sigma = 0{\cdot}73$, $\omega = 5$ and $a = 3{\cdot}117$. – – –, first approximation; ——, second approximation. S signifies 'synchronous' and H signifies 'half-frequency', for the curve below. S and H alternate for the curves to the right.

b/ω	1·0	2·5	5·0	10·0
0	1715·08	1715·08	1715·08	1715·08
0·5	2040	2037	2020	1976
0·75	2450	2403	2450	2354
1·0	2382	2352	2308	2411
1·25	2253	2227	2190	2130
1·5	2113	2098	2068	2005
2.5	1592	1600	1584	1526
4·5	1034	1059	1043	1021

TABLE 1. Variation of the critical Rayleigh number R_c with the fluctuation amplitude b and frequency ω for $\sigma = 0{\cdot}73$ (air); first approximation.

b	R_c	a_c
0	1715·08	3·117
0·5	1764	3·1
1·0	1951	3·0
2·5	4048	4·0
5·0	2650	3·8

TABLE 2. Variations of the critical Rayleigh number R_c and the critical wavenumber a_c with b at $\sigma = 7{\cdot}0$ (water) and $\omega = 10$; first approximation.

their R_c increases monotonically with ϵ (our b), whereas for $b > 0{\cdot}6$ our results show loops of alternating H curves and S curves. Possibly, for their temperature profile, R_c is monotonic with ϵ for $\epsilon < 1$. It is also possible that their calculation followed the initial S curve even after its intersection with an H curve. Up to $b = 0{\cdot}6$, our R_c increases with b (as does theirs with ϵ), and is of the same order of magnitude as their R_c. In comparing our R_c with theirs, it must be kept in mind that our temperature profile is different from theirs (since only their bottom boundary condition involves oscillating temperature, the top temperature being steady), and that our R_c should not be the same as theirs even if all the other parameters (ω, b, etc.) were equal in the two cases.

Our results given in table 1 can be compared with the results of Rosenblat & Tanaka (1971) given in their figures 1 and 2. Although for $b = 0{\cdot}5$ our R_c decreases monotonically with ω, in agreement with their figure 1 (for $\epsilon = 0{\cdot}4$), for $b = 1$ (corresponding to their $\epsilon = 1$) our R_c has a minimum near $\omega = 5$, whereas the R_c values shown in their figure 2 decrease monotonically with the frequency. Taking a look at our figures 2 and 3, we think the qualitative disagreement comes from the fact that at $b = 1$ we already encounter the second H curve, whereas in their case either $\epsilon = 1$ is still in the region of the first S curve (for their temperature profile) or in the calculation they have followed their first S curve beyond an intersection with an H curve.

In brief, our results do not necessarily contradict those of Rosenblat & Tanaka.

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